CONTACT PROBLEM OF TWO PRESTRESSED HALF-SPACES

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The complete solution of the problem of contact between two half-spaces distorted by the preliminary action of concentrated forces directed to different sides is known and used in the theory of cracks (see [1], e.g., where the plane variant is considered since the spatial variant is examined analogously). The problem of the contact between two half-spaces having a significantly more general distortion is considered in this paper: It is assumed that the spacing between the distorted half-spaces is a positive-homogeneous function at infinity (a negative degree of homogeneity). Although this problem does not indeed allow exact solution, all the qualitative deductions can be obtained that are analogous to the deductions of crack theory. These deductions are obtained from similarity considerations.

1. CONTACT BETWEEN AN ABSOLUTELY STIFF INFINITE STAMP AND AN ELASTIC HALF-SPACE

Let us consider infinite stamps whose surface is described by the function $x_3 = f(x_1, x_2)$. We assume that for sufficiently large x_1 and x_2, $x_1^2 + x_2^2 \geq \rho^2$, where ρ is some radius, the function $f(x_1, x_2)$ is determined by a positive smooth, positive-homogeneous function of the negative power β, i.e.,

$$f(x_1, x_2) > 0 \quad \forall (x_1, x_2) \in R^2, \qquad (1.1)$$
$$f(x_1, x_2) \in C^1(R^2 \backslash \{0\}),$$
$$\forall \lambda > 0, \quad f(\lambda x_1, \lambda x_2) = \lambda^\beta f(x_1, x_2), \quad \beta < 0 \quad \forall (x_1, x_2): \ \sqrt{x_1^2 + x_2^2} \geqslant \rho.$$

It is known [2] that all the stresses and displacement in the elastic half-space $x_3 \leq 0$ on whose boundary plane the tangential stresses σ_{i3} ($i = 1, 2$) are zero, are determined by a harmonic function F whose values for $x_3 = 0$ are related to u_3 and σ_{33} by the dependences

$$u_3 = 2(1 - \nu) F(x_1, x_2, 0), \quad \sigma_{33} = 2\mu \partial F(x_1, x_2, 0)/\partial x_3, \qquad (1.2)$$

where ν and μ are the Poisson ratio and the shear modulus of the half-space.

Let an absolutely stiff stamp be impressed translationally without friction into an elastic half-space. The contact problem is formulated as follows: For a given stamp shape $f(x_1, x_2)$ and an impressing stress S find a domain G on the boundary of a half-space at whose points contact will occur between the stamp and the half-space, the constant α that is the elastic closure between the bodies, and the harmonic function F in (1.2). The values of G, α, and F should satisfy the following conditions:

$$c_1 F(x_1, x_2, 0) = f(x_1, x_2) - \alpha, \quad (x_1, x_2) \in G \cup \partial G, \qquad (1.3)$$
$$\partial F(x_1, x_2, 0)/\partial x_3 = 0, \quad (x_1, x_2) \in R^2 \backslash G,$$
$$c_2 \partial F/\partial x_3 = S \text{ on } \infty \text{ for } x_3 = 0,$$

where $c_1 = 2(1 - \nu)$, $c_2 = 2\mu$; ∂G is the boundary of the open domain G.

<u>THEOREM</u>. Let the function $f(x_1, x_2)$ defining the stamp surface be the product of a function $f_0(x_1, x_2)$ and a parameter A, $A > 0$, $f = Af_0$, where f_0 is a positive, smooth, positive-homogeneous function of degree β ($\beta < 0$).

Let the harmonic function F_{11}, the domain G_{11}, and the constant α_{11} yield a solution of the contact problem (1.3) for the parameter $A = 1$ and the stress $S = 1$; then for an arbitrary parameter A and a stress S the solution of the contact problem will be given by the quantities F, α, and G defined by the following:

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$$F(x_1, x_2, x_3) = A\gamma^{-\beta}F_{11}(\gamma x_1, \gamma x_2, \gamma x_3), \ \alpha = A\gamma^{-\beta}\alpha_{11},$$
$$(x_1, x_2) \in G, \quad \text{if and only if} \quad (\gamma x_1, \gamma x_2) \in G_{11}, \tag{1.4}$$

where $\gamma = (S/A)^{1/(1-\beta)}$.

Proof. From the definition of the function F there follows that it is harmonic. Taking into account (1.4), (1.3), and (1.1), we obtain

$$c_1F(x_1, x_2, 0) = c_1A\gamma^{-\beta}F_{11}(\gamma x_1, \gamma x_2, 0) = A\gamma^{-\beta}f_0(\gamma x_1, \gamma x_2) - A\gamma^{-\beta}\alpha_{11} = Af_0(x_1, x_2) - \alpha = f(x_1, x_2) - \alpha,$$

i.e., the first of conditions (1.3) is satisfied.

Let us confirm compliance with the second and third of conditions (1.3):

$$\left.\frac{\partial F(x_1, x_2, 0)}{\partial x_3}\right|_{(x_1,x_2)\in R^2\setminus G} = A\gamma^{-\beta}\left.\frac{\partial F_{11}(\gamma x_1, \gamma x_2, 0)}{\partial (\gamma x_3)}\right|_{(x_1,x_2)\in R^2\setminus G}\gamma = A\gamma^{1-\beta}\left.\frac{\partial F_{11}(x_1, x_2, 0)}{\partial x_3}\right|_{(x_1,x_2)\in R^2\setminus G_{11}} = 0.$$

We analogously obtain

$$c_2\left.\frac{\partial F(x_1, x_2, 0)}{\partial x_3}\right|_{(x_1,x_2)\in G} = A\gamma^{1-\beta}c_2\left.\frac{\partial F_{11}(x_1, x_2, 0)}{\partial x_3}\right|_{(x_1,x_2)\in G_{11}} = A\gamma^{1-\beta} = S.$$

COROLLARY. If the solution of the contact problem (1.3) is known for any fixed values of the parameter A_1 and the load S_1, then by similarity it is known for any A and S for a stamp of the shape mentioned.

The following qualitative deductions can be made from the theorem.

Deductions. The following assertions hold for the impression of an absolutely stiff stamp whose shape $f(x_1, x_2)$ is determined by the function $f_0(x_1, x_2)$ and the positive parameter A, $f = Af_0$, where f_0 is a positive smooth positive-homogeneous function of degree β ($\beta < 0$), in an elastic half-space without friction.

1. The domain $R^2\setminus G$, where there is no contact, is obtained from the domain $R^2\setminus G_{11}$ by tension in all directions with a coefficient $\gamma^{-1} = A^{1/(1-\beta)}S^{1/(\beta-1)}$, i.e., the dimension a of the domain $R^2\setminus G$ is proportional to the quantity (A/S) to the power $1/(1-\beta)$:

$$a \sim A^{1/(1-\beta)}S^{1/(\beta-1)} \tag{1.5}$$

(the distance between the two most remote points of the domain can be taken as a).

2. Closure between the stamp and the half-space is proportional to the impressing stress S to the power $\beta/(\beta-1)$ and the parameter A to the power $1/(1-\beta)$:

$$\alpha \sim A^{1/(1-\beta)}S^{\beta/(\beta-1)}. \tag{1.6}$$

Indeed, it is seen from (1.4) that $\partial G = \{(x_1, x_2): (\gamma x_1, \gamma x_2) \in \partial G_{11}\}$, from which (1.5) is obtained easily.

The relationship (1.6) results directly from the second formula in the system (1.4)

Analogous deductions follow rapidly for the case of the classical problem about the contact of a convex stiff stamp with an elastic half-space. In formulating the appropriate theorem, the condition $\beta < 0$ must be replaced by the condition $\beta > 1$, and the contact-free domain $R^2\setminus G$ by the contact domain G [3].

2. FRICTIONLESS CONTACT OF TWO DISTORTED HALF-SPACES

Let us consider two infinite elastic bodies. We will mark the quantity referring to the first body by the superscript plus, and the quantity referring to the second body by the superscript minus. Let points of the first body lie above the plane $x_3 = 0$, and points of the second below.

Let there be a radius ρ such that for $\sqrt{x_1^2 + x_2^2} \geq \rho$ the distance $f(x_1, x_2)$ between the bodies will be determined by a positive, smooth, positive-homogeneous function of degree β ($\beta < 0$).

Let us start to impress these bodies by a pressure S applied at infinity. We consider that each of the bodies can be replaced by a half-space; then the contact problem reduces to seeking the harmonic functions F^+ and F^-, the domain G, and the constant α satisfying the conditions

$$2(1-\nu^+)F^+ + 2(1-\nu^-)F^- = f(x_1, x_2) - \alpha, \ (x_1, x_2) \in G \cup \partial G, \tag{2.1}$$
$$\partial F^\pm(x_1, x_2, 0)/\partial x_3 = 0, \ (x_1, x_2) \in R^2 \setminus G,$$
$$2\mu^\pm \partial F^\pm/\partial x_3 = S \text{ on } \infty \text{ for } x_3 = 0,$$
$$\mu^+ \partial F^+(x_1, x_2, 0)/\partial x_3 = \mu^- \partial F^-(x_1, x_2, 0)/\partial x_3, \ (x_1, x_2) \in G.$$

It is known that the contact problem (2.1) reduces to the contact problem (1.3) with the new constants c_1 and c_2 if the function giving the shape of the stamp is replaced by a function giving the distance between the body surfaces. Then, in the case of frictionless contact between two elastic bodies, between whose surfaces the distance $f(x_1, x_2)$ is determined by a positive, smooth, positive-homogeneous function of the negative degree β, the theorem in Sec. 1, its corollary, and deductions are valid.

As an illustration, let us consider the problem [1] in which two elastic half-spaces with absolutely smooth boundaries contiguous on the plane $0x_1x_2$ are preliminarily deformed by two concentrated forces P applied to each of the half-spaces at the origin 0 and are squeezed together by a stress S orthogonal to the plane $0x_1x_2$ at infinity. Determine to what degrees of P and S is the linear dimension a of the gap being formed between the half-spaces proportional under the condition that adhesion forces are completely absent on the half-space boundary.

It follows from the solution of the Boussinesq problem [2] that the normal displacements of points of the boundary are expressed in the form

$$u_3^\pm = \frac{1-\nu^\pm}{2\pi\mu^\pm}\frac{P}{r}, \text{ where } r = \sqrt{x_1^2 + x_2^2}.$$

It hence follows that prior to application of the stress S the distance between the body boundaries $f(x_1, x_2)$ was given by a positive, smooth, positive-homogeneous function of degree minus one

$$f(x_1, x_2) = \left(\frac{1-\nu^+}{2\pi\mu^+} + \frac{1-\nu^-}{2\pi\mu^-}\right)\frac{P}{r}.$$

Then the conditions of the theorem are valid. We obtain from (1.5) that the radius of the gap between the half-spaces is proportional to the square root of P/S, namely, $a \sim P^{1/2}S^{-1/2}$.

This result agrees with the result in the theory of cracks, obtained during examination of a disk-shaped crack at whose center rupturing concentrated forces are applied while a compressive stress acts at infinity. Writing the expression for the stress intensity factor for such a crack, and equating it to zero, we obtain an equation in a. The method of solving this problem by the theory of cracks is described in [1].

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LITERATURE CITED

1. L. I. Sedov, Mechanics of a Continuous Medium [in Russian], Vol. 2, Nauka, Moscow (1976).
2. Yu. N. Rabotnov, Mechanics of a Deformable Solid [in Russian], Nauka, Moscow (1979).
3. F. M. Borodich, "Similarity in the problem of the contact of two elastic bodies," Prikl. Mat. Mekh., 47, No. 3 (1983).